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Correlators of left charges and weak operators in finite volume chiral perturbation theory

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Abstract: We compute the two-point correlator between left-handed flavour charges, and the three-point correlator between two left-handed charges and one strangeness violating $\Delta I = 3/2$ weak operator, at next-to-leading order in finite volume $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ chiral perturbation theory, in the so-called ϵ-regime. Matching these results with the corresponding lattice measurements would in principle allow to extract the pion decay constant F, and the effective chiral theory parameter g_{27}, which determines the $\Delta I = 3/2$ amplitude of the weak decays $K \to \pi\pi$ as well as the kaon mixing parameter $\hat{B}_K$ in the chiral limit. We repeat the calculations in the replica formulation of quenched chiral perturbation theory, finding only mild modifications. In particular, a properly chosen ratio of the three-point and two-point functions is shown to be identical in the full and quenched theories at this order.

Keywords: Weak Decays, Kaon Physics, Lattice QCD, Chiral Lagrangians.

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1. Introduction

The study of the weak matrix elements involved in kaon physics is a long-standing topic of lattice QCD [1, 2]. It is however a very difficult problem for all lattice formulations which break the chiral symmetry explicitly, such as Wilson fermions. We expect that significant progress can be achieved with the new Ginsparg-Wilson formulations of lattice fermions [3]–[10], which possess an exact chiral symmetry in the limit of vanishing quark masses, resulting in an enormous simplification in weak operator mixing and renormalization [8], [11]–[13].

One advantage of approaching the regime of vanishing quark masses is obviously that the uncertainty induced by chiral extrapolations is avoided. On the other hand, since the chiral limit implies that light mesons become massless, it necessarily brings with it large finite volume effects. This apparent difficulty can, however, be turned into a useful tool: if the finite volume effects can be resolved analytically in terms of the infinite volume properties of the theory, then the infinite volume properties can be extracted by monitoring the volume dependence. The first proposal to apply finite-size scaling techniques to the weak $K \to \pi\pi$ amplitudes was presented in [14].

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A practical realisation for the finite-volume philosophy mentioned is offered by Chiral Perturbation Theory (χPT). As the quark masses get smaller, the chiral expansion becomes more and more accurate in describing the dynamics of the low momentum modes of QCD (below a few hundred MeV). The chiral expansion in this regime is slightly more complicated than in infinite volume, because it requires the resummation of pion zero mode contributions. Gasser and Leutwyler [15] have presented a systematic procedure for doing this, the so-called ϵ-expansion (see also [16]). Several observables, such as the quark condensate and the scalar and vector two-point functions, have already been computed at next-to-leading order in the ϵ-expansion [15]–[18]. These quantities depend on the (infinite volume) chiral theory parameters, the volume, and the quark masses, in a way that a comparison with lattice data for different volumes and quark masses, allows in principle the extraction of the corresponding infinite volume low-energy couplings of χPT.

Obviously the chiral model can be extended to include the $|\Delta S| = 1$ weak hamiltonian [19]. This introduces a new set of low-energy constants, from which the physical amplitudes in kaon decays can be extracted, by working up to some desired order in the chiral expansion. The determination of these low-energy constants by matching the matrix elements of weak operators computed in lattice QCD to the same observables computed in $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ χPT was proposed a long time ago [2]. Calculations along these lines (for a recent review, see [20]) have shown that if the matching is performed at relative large quark masses, there are large uncertainties induced by the chiral extrapolations [21], due to the fact that next-to-leading order corrections involve a large number of new unknown couplings [22]. Although strategies have been proposed [23] to measure the relevant new couplings, together with the leading order ones, by matching the matrix elements at several kinematical conditions, this is clearly a very challenging procedure, particularly for the $\Delta I = 1/2$ kaon decays [23, 24].

The approach that we consider here is instead to perform the matching in a finite volume but close to the chiral limit, in the ϵ-regime. The predictions of χPT for the weak matrix elements in terms of the low-energy couplings are then not the same as in infinite volume. We compute the correlators of two left-handed flavour currents, as well as the matrix elements of the $|\Delta S| = 1$ (or $|\Delta S| = 2$) weak hamiltonian with two such currents, at next-to-leading order in the ϵ-expansion. Some motivations for this approach have been discussed in [13]. The simplification brought in by approaching the chiral limit is manifest in the fact that at next-to-leading order none of the unknown couplings present in the usual p-expansion [22] contributes. We expect therefore that the determination of the leading order couplings should in principle be more straightforward. Only the 27-plet low-energy coupling contributing to $\Delta I = 3/2$ kaon decays is considered here, due to a number of subtleties with the octet operator [23, 24], which will be considered elsewhere.

The structure of the paper is as follows. In section 2, we discuss the chiral model representation of the weak hamiltonian at low energies in the $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetric case. In section 3, we review the ϵ-expansion of Gasser and Leutwyler for this model, and in section 4 present the results of our next-to-leading order calculations. Finally, in section 5, we present the same results in the quenched theory. We conclude in section 6.

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2. Weak operators in the SU(3) chiral theory

Ignoring weak interactions, the QCD chiral lagrangian possesses an $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry, broken "softly" by the mass terms. The Euclidean lagrangian can to leading order in a momentum expansion be written as

$$\mathcal{L}_E = \frac{F^2}{4} \mathrm{Tr}\left[\partial_\mu U \partial_\mu U^\dagger\right] - \frac{\Sigma}{2} \mathrm{Tr}\left[U M e^{i\theta/N_f} + M^\dagger U^\dagger e^{-i\theta/N_f}\right]. \tag{2.1}$$

Here $U \in \mathrm{SU}(3)$, θ is the vacuum angle, $N_f = 3$, M is the quark mass matrix and, to leading order in the chiral expansion, F, Σ, equal the pseudoscalar decay constant and the chiral condensate, respectively. We shall for convenience take M to be real and diagonal.

Weak interactions break explicitly the $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry of eq. (2.1). In the fundamental theory, the strangeness violating interactions responsible for kaon decays can be accurately described through an operator product expansion in the inverse W-boson mass. In the CP conserving case of two generations, the effective weak hamiltonian to leading order in the QCD coupling constant is then (see, e.g., [25, 26])

$$H_w = 2\sqrt{2} G_F V_{ud} V_{us}^* \left[(\bar{s}\gamma_\mu P_- u)(\bar{u}\gamma_\mu P_- d) - (\bar{s}\gamma_\mu P_- c)(\bar{c}\gamma_\mu P_- d)\right] + \mathrm{h.c.}, \tag{2.2}$$

where G_F is the Fermi constant, V_{ij} are elements of the CKM-matrix, and $P_\pm = (1 \pm \gamma_5)/2$. Taking into account QCD radiative corrections, the coefficients get modified, but H_w can still be written as [27, 12]

$$H_w = 2\sqrt{2} G_F V_{ud} V_{us}^* \left\{ \sum_{\sigma=\pm 1} h_w^\sigma \Big([O_w]^\sigma_{suud} - [O_w]^\sigma_{sccd} \Big) + h_m [O_m]_{sd} \right\} + \mathrm{h.c.}, \tag{2.3}$$

where $h_w^\pm, h_m$ are regularisation dependent dimensionless Wilson coefficients, and we have introduced the notation

$$[O_w]^\sigma_{rsuv} \equiv \frac{1}{2}\Big([O_w]_{rsuv} + \sigma [O_w]_{rsvu} \Big), \tag{2.4}$$

$$[O_w]_{rsuv} \equiv (\bar{\psi}_r \gamma_\mu P_- \psi_u)(\bar{\psi}_s \gamma_\mu P_- \psi_v), \tag{2.5}$$

$$[O_m]_{sd} \equiv (m_c^2 - m_u^2)\{[\bar{\psi} M]_s P_- \psi_d + \bar{\psi}_s P_+ [M^\dagger \psi]_d\}. \tag{2.6}$$

Here r, s, u, v are generic flavour indices, while u, d, s, c denote the physical flavours. The Wilson coefficients have been computed also for Ginsparg-Wilson "overlap" fermions [12], apart from h_m, which remains undetermined. According to eq. (2.2), the leading order values are $h_w^\pm = 1, h_m = 0$.

In order to match the hamiltonian of eq. (2.3) to the one in the SU(3) chiral theory, the first step is to decompose it into irreducible representations of the $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ flavour group, present at low energies. For completeness, we review the general formulae for the decomposition in appendix A. The weak operators are singlets under $\mathrm{SU}(3)_R$, and denoting projected operators transforming under representations of $\mathrm{SU}(3)_L$ with dimensions 27, 8 by $[\hat{O}_w]^+_{rsuv}$, $[R_w]^\sigma_{ru}$, respectively, the weak hamiltonian can be rewritten as

$$\begin{aligned} H_w = 2\sqrt{2} G_F V_{ud} V_{us}^* \Big\{ & h_w^+ [\hat{O}_w]^+_{suud} + \frac{1}{5} h_w^+ [R_w]^+_{sd} - h_w^- [R_w]^-_{sd} - \frac{1}{2}(h_w^+ + h_w^-)[O_w]_{sccd} - \\ & - \frac{1}{2}(h_w^+ - h_w^-)[O_w]_{scdc} + h_m [O_m]_{sd} \Big\} + \mathrm{h.c.}, \end{aligned} \tag{2.7}$$

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where

$$[\hat{O}_w]^+_{suud} \equiv \frac{1}{2}\Big\{[O_w]_{suud} + [O_w]_{sudu} - \frac{1}{5}\sum_{k=u,d,s}\Big([O_w]_{skdk} + [O_w]_{skkd}\Big)\Big\}\,, \tag{2.8}$$

$$[R_w]^\pm_{sd} \equiv \frac{1}{2}\sum_{k=u,d,s}\Big([O_w]_{skdk} \pm [O_w]_{skkd}\Big)\,. \tag{2.9}$$

The first operator in eq. (2.7) transforms under the 27-plet of the $\mathrm{SU}(3)_L$ subgroup: it is symmetric under the interchange of quark or antiquark indices, and traceless. The remaining ones, transforming as $\mathbf{3}^* \otimes \mathbf{3}$ and being traceless, belong to irreducible representations of dimension 8.

The next step is to find the chiral analogue for this weak hamiltonian, as well as for the left-handed flavour currents, which we will use as external probes. In a convention for fermion fields where the Euclidean lagrangian reads ($\gamma_\mu^\dagger = \gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$)

$$L_E = \bar{\psi}(\gamma_\mu D_\mu + MP_- + M^\dagger P_+)\psi\,, \tag{2.10}$$

we may define a left-handed current as

$$\Big(J^a_\mu\Big)_{\mathrm{QCD}} \equiv i\bar{\psi}_r T^a_{ru}\gamma_\mu P_- \psi_u\,, \tag{2.11}$$

where the T^a are Hermitean generators of the flavour SU(3).[1] As usual, it is convenient to introduce an external left-handed flavour gauge field source, A^a_μ, such that

$$\Big(J^a_\mu\Big)_{\mathrm{QCD}} = \frac{\partial L_E}{\partial A^a_\mu}\,. \tag{2.12}$$

Observables including $(J^a_\mu)_{\mathrm{QCD}}$ can then be addressed within the chiral theory by coupling also the pion field covariantly to A^a_μ, and taking functional derivatives with respect to it. More concretely, the partial derivatives of eq. (2.1) are promoted to covariant ones,

$$\partial_\mu U \to D_\mu U \equiv [\partial_\mu + iA^a_\mu T^a]U\,, \tag{2.13}$$

and the left-handed current is defined as

$$\Big(J^a_\mu\Big)_{\chi\mathrm{PT}} \equiv \mathcal{J}^a_\mu \equiv \Big(\frac{\partial \mathcal{L}_E}{\partial A^a_\mu}\Big)\Big|_{A^a_\mu=0} = -i\frac{F^2}{2}T^a_{ru}\Big(\partial_\mu U U^\dagger\Big)_{ur}\,, \tag{2.14}$$

up to higher order corrections.

We can now find the chiral analogues for the building blocks in eqs. (2.4)–(2.6). By a comparison of eqs. (2.5), (2.11), the analogue of the operator in eq. (2.5), denoted in the chiral case by $[\mathcal{O}_w]_{rsuv}$, can be written as

$$[\mathcal{O}_w]_{rsuv} = \frac{1}{4}F^4\Big(\partial_\mu U U^\dagger\Big)_{ur}\Big(\partial_\mu U U^\dagger\Big)_{vs}\,. \tag{2.15}$$

[1] In fact, the only property of T^a we need to assume is their tracelessness.

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The operator of eq. (2.15) is the only one with the same symmetry properties as its counterpart in the fundamental theory, at the leading order in the chiral expansion. Correspondingly, we can also find the chiral counterpart for the operator O_m, by employing scalar and pseudoscalar external sources S^a, P^a, defined with the substitution $M \to M + S^a T^a - iP^a T^a$, and taking derivatives with respect to the sources:

$$[\mathcal{O}_m]_{sd} = -(m_c^2 - m_u^2)\frac{\Sigma}{2}(UMe^{i\theta/N_f} + M^\dagger U^\dagger e^{-i\theta/N_f})_{ds} \,. \tag{2.16}$$

Given these building blocks, we can determine all the operators (at the leading order in the chiral expansion) transforming under the 27-plet and octet of $\mathrm{SU}(3)_L$, which allows then to translate the weak hamiltonian of eq. (2.7) to the chiral theory. Because some contractions are traceless, cf. eqs. (A.7), (A.8), there are only three such operators [2]. For the physical choice of indices, we write these as

$$\mathcal{O}_{27} \equiv [\hat{\mathcal{O}}_w]^+_{suud} = \frac{3}{5}\Big([\mathcal{O}_w]_{sudu} + \frac{2}{3}[\mathcal{O}_w]_{suud}\Big)\,, \tag{2.17}$$

$$\mathcal{O}_8 \equiv [\mathcal{R}_w]^+_{sd} = \frac{1}{2}\sum_{k=u,d,s}[\mathcal{O}_w]_{skkd}\,, \tag{2.18}$$

$$\mathcal{O}'_8 \equiv \frac{1}{2}F^2\Sigma(UMe^{i\theta/N_f} + M^\dagger U^\dagger e^{-i\theta/N_f})_{ds}\,, \tag{2.19}$$

where we have made use of eqs. (A.7), (A.8) to simplify the chiral versions of eqs. (2.8), (2.9). Note that in the definition of $\mathcal{O}'_8$ here we have left out the explicit mass combination $(m_c^2 - m_u^2)$, which can then appear in the coefficient of this operator; the coefficient can, however, also receive other contributions, due to mixings with operators of the same symmetries.

We can now write down the analogue of H_w in eq. (2.7) in the chiral theory. We denote it by $\mathcal{H}_w$. To again define dimensionless coefficients, we write $\mathcal{H}_w$ in the form

$$\mathcal{H}_w \equiv 2\sqrt{2}G_F V_{ud}V^*_{us}\left\{\frac{5}{3}g_{27}\mathcal{O}_{27} + 2g_8\mathcal{O}_8 + 2g'_8\mathcal{O}'_8\right\} + \text{h.c.}\,, \tag{2.20}$$

where g_{27}, g_8 and g'_8 are the low-energy constants we are interested in [28].

Now, it is easy to see that the amplitude for $\Delta I = 3/2$ decays, such as $K^\pm \to \pi^\pm\pi^0$, is directly proportional to g_{27}, while the much faster $\Delta I = 1/2$ decays of K^0_S get a comparable contribution both from g_8 and g_{27}. (The parameter g'_8, on the other hand, does not contribute to physical kaon decays [27, 2, 29].) More quantitatively, a leading order analysis in infinite volume [28], supplemented by phenomenologically determined large phase shifts [30] in the amplitudes, suggests the well-known values

$$|g_{27}| \approx 0.29\,, \tag{2.21}$$

$$|g_8| \approx 5.1\;. \tag{2.22}$$

It has been argued that 1-loop corrections in the chiral perturbation theory are large [22, 31, 32], and one can therefore get agreement with experimental data on partial decay widths even with somewhat less differing values of g_8 and g_{27}, but a hierarchy still remains.

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In the limit $N_c \to \infty$, on the other hand, one obtains [28] the "tree-level" values deducible from the naive conversion of eq. (2.7), with $h_w^\pm$ as in eq. (2.2), to the corresponding chiral operators of eqs. (2.17), (2.18):

$$g_{27} = g_8 = \frac{3}{5}\,. \tag{2.23}$$

Clearly a large non-perturbative enhancement of g_8 with respect to the tree-level value, and some reduction of g_{27}, is needed to fit the experiment. The final goal is to improve on the naive estimates in eq. (2.23), by determining g_{27} and g_8 non-perturbatively in the $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetric theory. As mentioned in the introduction, we will in this paper discuss only g_{27}, due to various subtleties in the determination of g_8 (particularly in the quenched case).

Let us finally recall that another physical observable determined by g_{27} is the $\hat{B}_K$, characterising the mixing of $K^0, \bar{K}^0$, and hence determining the mass difference of K_S, K_L [33]. It is defined by[2]

$$\langle\, \bar{K}^0 \,|\, h_{\Delta S=2}\, [O_w]_{ssdd} \,|\, K^0 \,\rangle \equiv \frac{4}{3}(m_K F_K)^2 \hat{B}_K\,, \tag{2.24}$$

where $h_{\Delta S=2}$ is the Wilson coefficient (see, e.g., [34] and references therein) related to the operator $[O_w]_{ssdd}$, normalised to unity at tree-level. Since $[O_w]_{ssdd}$ is symmetric and traceless, it belongs to the 27-plet. Therefore, if $h_{\Delta S=2}$ is replaced by h_w^+ in eq. (2.24), the matrix element is in the chiral limit (where $m_K = m_\pi = 0, F_K = F_\pi = F$) proportional to g_{27}:

$$\frac{4}{3}\hat{B}_K = \frac{h_{\Delta S=2}}{h_w^+} \cdot \frac{5}{3} g_{27}\,. \tag{2.25}$$

The tree-level value is then $\hat{B}_K = 3/4$, but going to next-to-leading order in the large-N_c approach one finds a suppression down to $\hat{B}_K \simeq 0.3 \ldots 0.4$ [35]. This suppression factor is very close to what would be needed for g_{27} to go from eq. (2.23) to eq. (2.21). We may thus consider it a further motivation for the lattice study to corroborate this prediction of the large-N_c approach. Note that near the physical point ($m_K > 0$) a considerably larger value is found (for recent reviews, see [36]).

3. Chiral perturbation theory in a finite volume

In order to determine g_{27}, we will consider the lattice measurement of left-current two-point correlation functions in the fundamental theory at low enough momenta [13]. In this regime we assume that the effective theory gives a good description, and thus require, to first order in the weak hamiltonian,

$$\frac{\delta^2}{\delta A_\mu^a(x)\delta A_\nu^b(y)}\Big\langle \int_z H_w(z) \Big\rangle_{\mathrm{QCD}} = \frac{\delta^2}{\delta A_\mu^a(x)\delta A_\nu^b(y)}\Big\langle \int_z \mathcal{H}_w(z) \Big\rangle_{\chi\mathrm{PT}}\,, \tag{3.1}$$

where A_μ^a is set to zero after the differentiations. The measurement of the left-hand-side in lattice QCD allows to tune the effective couplings in the weak hamiltonian appearing

[2] The parameter B_K is defined identically to $\hat{B}_K$ but without the Wilson coefficient $h_{\Delta S=2}$, whereby it is scheme and scale dependent.

on the right-hand-side. In general, it is convenient even to remove the integral $\int_z (\ldots)$ appearing in eq. (3.1), since matching can also be achieved before this averaging, as long as the currents brought down by the functional derivatives are far enough from each other, and $H_w(z)$. In this way we avoid complications with "contact terms", arising from operators overlapping at the same spacetime location. It is also convenient to consider space-averaged charges positioned at different times, x_0, y_0, rather than local currents.

We thus discuss the product of two left-handed charges separated from the weak hamiltonian sitting at the origin, $z \equiv 0$. To keep the discussion as general as possible, we consider matrix elements of the "unprojected" operator $[\mathcal{O}_w]_{rsuv}$ in eq. (2.15); a projection to the actual 27-plet $[\hat{\mathcal{O}}_w]^{+}_{rsuv}$ is then carried out by the operations in appendix A. Thus, writing the expressions in a form where their QCD analogues are obvious,[3] we will be concerned with

$$\mathcal{C}^{ab}(x_0) \equiv \int_{\mathbf{x}} \langle \mathcal{J}^a_0(x) \mathcal{J}^b_0(0) \rangle \,, \tag{3.2}$$

$$[\mathcal{C}_w]^{ab}_{rsuv}(x_0, y_0) \equiv \int_{\mathbf{x}} \int_{\mathbf{y}} \langle \mathcal{J}^a_0(x) [\mathcal{O}_w]_{rsuv}(0) \mathcal{J}^b_0(y) \rangle \,, \tag{3.3}$$

where $\int_{\mathbf{x}} = \int \mathrm{d}^3 x$. The computations are carried out with a finite phase θ as in eq. (2.1), allowing to make predictions for the case of a fixed topology, as well (see section 4.3).

Due to the vicinity of the chiral limit, we take the volume to be a finite periodic box, of size $V = L_0 L_1 L_2 L_3$. Momenta are then quantised,

$$p_\mu = \frac{2\pi}{L_\mu} n_\mu \,, \qquad n_\mu \in \mathbb{Z} \,. \tag{3.4}$$

Since we want to be close to the chiral limit in the finite volume, the computation is organised according to the rules of the ϵ-expansion [15]. In the ϵ-expansion one writes

$$U = \exp\left(i \frac{2\xi}{F} \right) U_0 \,, \tag{3.5}$$

where ξ has non-zero momentum modes only, while U_0 is a constant SU(3) matrix collecting the zero modes. The integration over U_0 has to be carried out exactly when $m\Sigma V \lesssim \mathcal{O}(1)$, where m is a quark mass, while the integration over the non-zero modes can be carried out perturbatively as long as $FL \gg 1$. The power counting rules for the ϵ-expansion are

$$F \sim \mathcal{O}(1) \,, \qquad \partial_\mu \sim \mathcal{O}(\epsilon) \,, \qquad L_\mu \sim \mathcal{O}(1/\epsilon) \,, \qquad \xi \sim \mathcal{O}(\epsilon) \,, \qquad m \sim \mathcal{O}(\epsilon^4) \,. \tag{3.6}$$

Note that the quark mass counts as four powers of the momenta, rather than two as in the standard chiral expansion in infinite volume, where $m \sim M_\pi^2 \sim \partial_\mu^2$. The perturbative integrals for the non-zero momentum modes are computed with the measure

$$\int_{p'} \equiv \frac{1}{V} \sum_{\{n_\mu\}} \left(1 - \delta^{(4)}_{n,0} \right) . \tag{3.7}$$

[3] At the present order, these forms differ from those obtained by taking functional derivatives with respect to a flavoured gauge field, as in eq. (3.1), only regarding unimportant "contact terms" $\sim \delta(x_0)$, $\delta(y_0)$, $\delta(x_0 - y_0)$.

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T/L	β_1	k_{00}
32/32	0.14046	$0.07023 = \beta_1/2$
32/28	0.13872	0.07826
32/24	0.13215	0.08186
32/20	0.11689	0.08307
32/16	0.08360	0.08331

Table 1: Some numerical values for β_1, k_{00}, defined in eqs. (3.10), (3.11), for geometries of the type $L_0 \equiv T, L_1 = L_2 = L_3 \equiv L$.

We will compute at next-to-leading order in the ϵ-expansion, including corrections of relative order $\mathcal{O}(\epsilon^2)$. It turns out that the physical pion decay constant and mass, F_π, M_π, differ from their leading order values, $F, 2m\Sigma/F^2$ (for $M = \mathrm{diag}(m,m,m)$), only by terms of relative order $\mathcal{O}(\epsilon^4)$ [18], an effect which may thus be ignored. This is a result of the fact that no higher order operators in the action (i.e., none of the L_i's of Gasser and Leutwyler) contribute at $\mathcal{O}(\epsilon^2)$. Similarly, there are no higher order operators of $\mathcal{O}(\epsilon^2)$ in the chiral representation of $[O_w]_{rsuv}$ in eq. (2.15). This is also in contrast with the usual chiral expansion in infinite volume, where a plethora of new operators contribute at next-to-leading order [22]. The fact that the contamination from higher order operators is small, is simply a consequence of being closer to the chiral limit.

The propagator for the non-zero momentum modes ξ follows by expanding the parametrisation in eq. (3.5) in ϵ, and inserting into eq. (2.1):

$$\Big\langle \xi_{ur}(x)\, \xi_{vs}(y) \Big\rangle = \frac{1}{2}\Big[\delta_{us}\delta_{vr} G(x-y) - \delta_{ur}\delta_{vs} E(x-y)\Big] , \tag{3.8}$$

where

$$G(x) = \int_{p'} \frac{e^{ip\cdot x}}{p^2} . \tag{3.9}$$

In the chiral case, $E(x) = G(x)/N_f$, but we keep everywhere $E(x)$ completely general. The reason is that then the form of eq. (3.8) is general enough to contain also the propagator of the replica formulation of quenched chiral perturbation theory [37, 38], and thus we can already include the main ingredients needed in section 5.

Where encountered, ultraviolet divergences are treated with dimensional regularization. For later reference let us define [39, 17], in particular, the two integrals appearing in the computation (see also [40]):

$$\int_{p'} \frac{1}{p^2} \equiv -\frac{\beta_1}{V^{1/2}} , \tag{3.10}$$

$$\int_{p'} \left(\frac{2p_0^2}{(p^2)^2} - \frac{1}{p^2} \right) = L_0 \frac{\mathrm{d}}{\mathrm{d}L_0} \int_{p'} \frac{1}{p^2} \equiv \frac{L_0}{L_1 L_2 L_3} k_{00} . \tag{3.11}$$

Here β_1, k_{00} are finite dimensionless numerical coefficients depending on the geometry of the box. Some values for them are listed in table 1.

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Figure 1: The $\mathcal{O}(\epsilon^2)$ graphs computed in section 4.1. An open box denotes a current (eq. (2.14)), a cross a "measure term" (cf. ref. [15]), and a filled circle a mass insertion.

4. Results

4.1 Charge — charge expectation value

We now proceed to apply the rules of the ϵ-expansion to the correlator $\mathcal{C}^{ab}(x_0)$, defined in eq. (3.2). The result can in fact also be inferred from [17], by summing together the expressions for the axial and vector flavour currents.

The graphs contributing to $\mathcal{C}^{ab}(x_0)$ are shown in figure 1. Apart from the graph including the mass insertion, current conservation guarantees that the result is independent of x_0. Indeed, we obtain

$$\begin{aligned}\mathcal{C}^{ab}(x_0) \;=\; & \Big(-\operatorname{Tr} T^a T^b\Big)\frac{F^2}{2L_0}\times \\ & \times\left\{1+\frac{N_f}{F^2}\left[\frac{\beta_1}{V^{1/2}}-\frac{L_0^2 k_{00}}{V}\right]+\frac{2\Sigma L_0^2}{N_f F^2}\left\langle \operatorname{Re}\operatorname{Tr}\left[M U_0 e^{i\theta/N_f}\right]\right\rangle_{\theta,U_0} h_1\Big(\frac{x_0}{L_0}\Big)\right\},\end{aligned} \tag{4.1}$$

where β_1, k_{00} are from eqs. (3.10), (3.11); and [39]

$$h_1(\tau) = \frac{1}{2}\left[\left(|\tau|-\frac{1}{2}\right)^2-\frac{1}{12}\right]. \tag{4.2}$$

Finally, $\langle\cdots\rangle_{\theta,U_0}$ denotes an average over the zero-mode Goldstone manifold,

$$\langle\cdots\rangle_{\theta,U_0} \equiv \frac{\int_{U_0}(\cdots)\exp(V\Sigma\operatorname{Re}\operatorname{Tr}\left[MU_0 e^{i\theta/N_f}\right])}{\int_{U_0}\exp(V\Sigma\operatorname{Re}\operatorname{Tr}\left[MU_0 e^{i\theta/N_f}\right])}\,, \tag{4.3}$$

where $\int_{U_0}$ is an integration over $\mathrm{SU}(N_f)$ according to the Haar measure.

As a simple explicit example, let us assume a box with $L_0 = L_1 = L_2 = L_3$, a mass matrix $M = \mathrm{diag}(m,m,m)$, and a phase $\theta = 0$. Taking furthermore into account that at the present order $\mathcal{O}(\epsilon^2)$, $F_\pi = F$, $M_\pi^2 = 2m\Sigma/F^2$, the full result may be expressed as

$$\mathcal{C}^{ab}(x_0) = \Big(-\operatorname{Tr} T^a T^b\Big)\frac{F_\pi^2}{2L_0}\left\{1+\frac{1}{F_\pi^2 L_0^2}\left[\frac{3}{2}\beta_1 + u\,\Sigma_{\theta=0}\Big(\frac{u}{2}\Big)\,h_1\Big(\frac{x_0}{L_0}\Big)\right]\right\}, \tag{4.4}$$

where [15, 17, 18]

$$u = M_\pi^2 F_\pi^2 L_0^4\,, \tag{4.5}$$

$$\Sigma_\theta(u/2) = \frac{1}{N_f}\left\langle \operatorname{Re}\operatorname{Tr}\,U_0 e^{i\theta/N_f}\right\rangle_{\theta,U_0} = \frac{2}{N_f}\frac{\partial}{\partial u}\ln\int_{U_0} e^{(u/2)\operatorname{Re}\operatorname{Tr}\,U_0\exp(i\theta/N_f)}\,, \tag{4.6}$$

$$\Sigma_{\theta=0}(u/2) \approx \begin{cases} u/(4N_f)\,, & u \ll 1 \\ 1, & u \gg 1 \end{cases}. \tag{4.7}$$

Note that in the notation of [18], $u\,\Sigma_{\theta=0}(u/2) = u^2 I_1(u)/(4N_f)$.

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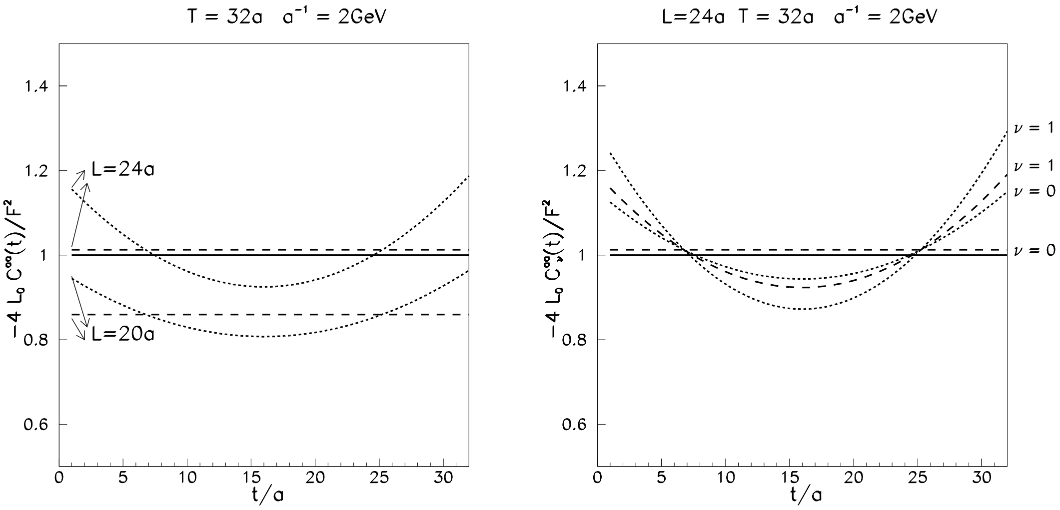

Figure 2: Left: The expression inside the curly brackets in eq. (4.1), for $L_0 = T$, $V = TL^3$, $a^{-1} = 2\,\mathrm{GeV}$, and $m = 0\,\mathrm{MeV}$ (dashed), $m = 5\,\mathrm{MeV}$ (dotted). We have assumed $\theta = 0$, $F = 93\,\mathrm{MeV}$, $\Sigma = (250\,\mathrm{MeV})^3$. The solid line is the tree-level result. Right: The same observable, in an ensemble with a fixed topological charge $\nu = 0, 1$ (cf. section 4.3), for $L = 24a$.

For $N_f = 3$, a numerical determination of $I_1(u)$ has been given in [18]. Using this result, we show in figure 2 examples for two asymmetric lattices, $32 \times 24^3, 32 \times 20^3$, and a lattice spacing $a^{-1} = 2\,\mathrm{GeV}$. The function in eq. (4.1) has been normalised to its (constant) tree-level value. The next-to-leading order correction is observed to become dangerously large for $L/a \lesssim 20$ (i.e., $L \lesssim 2$ fm).

4.2 Charge — charge — weak operator $\mathcal{O}_w$

We then move to $[\mathcal{C}_w]^{ab}_{rsuv}(x_0, y_0)$, defined in eq. (3.3). To this end we evaluate the graphs in figure 3. Let us mention that we are ignoring disconnected diagrams, since they only lead to trace parts, $\sim \delta^{ab}\delta_{ur}\delta_{vs}$, $\delta^{ab}\delta_{us}\delta_{vr}$, which would vanish in any case after the projection to $[\hat{\mathcal{O}}_w]^+_{rsuv}$. The non-trivial flavour structures arising from figure 3 always appear in one of the two combinations,

$$[\Delta^{(1)}]^{ab}_{rsuv} \equiv T^a_{ur}T^b_{vs} + T^a_{vs}T^b_{ur} \equiv T^{\{a}_{ur}T^{b\}}_{vs}\,, \tag{4.8}$$

$$[\Delta^{(2)}]^{ab}_{rsuv} \equiv [\Delta^{(1)}]^{ab}_{rsvu} - \frac{1}{2}\Big(\delta_{us}\{T^a, T^b\}_{vr} + \delta_{vr}\{T^a, T^b\}_{us}\Big)\,. \tag{4.9}$$

After lengthy but straightforward algebra, we obtain[4]

$$\begin{aligned}[\mathcal{C}_w]^{ab}_{rsuv}(x_0, y_0) = -\frac{F^4}{4L_0^2}\Big\{&\Delta^{(1)} + \Big(N_f\Delta^{(1)} + \Delta^{(2)}\Big)\frac{2}{F^2}\left[\frac{\beta_1}{V^{1/2}} - \frac{L_0^2 k_{00}}{V}\right] + \\ &+ \Delta^{(1)}\frac{2\Sigma L_0^2}{N_f F^2}\Big\langle \mathrm{Re}\,\mathrm{Tr}\,[MU_0e^{i\theta/N_f}]\Big\rangle_{\theta,U_0}\left[h_1\Big(\frac{x_0}{L_0}\Big) + h_1\Big(\frac{y_0}{L_0}\Big)\right]\Big\}\,,\end{aligned} \tag{4.10}$$

[4]The full expression before the volume average $\int_{\mathbf{x}}(\ldots)$ can also be obtained from the authors on request.

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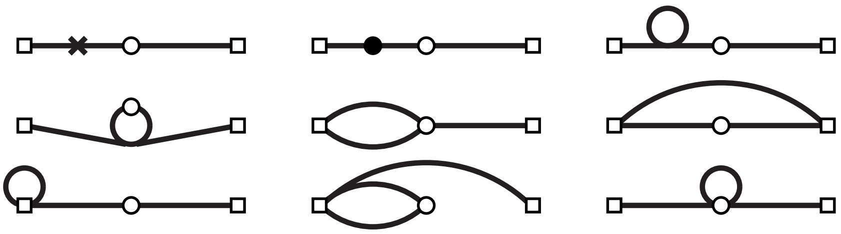

Figure 3: The $\mathcal{O}(\epsilon^2)$ graphs computed in section 4.2. An open circle denotes the weak operator in eq. (2.15), otherwise the notation is as in figure 1.

where we have for clarity omitted the indices from $[\Delta^{(1)}]^{ab}_{rsuv}, [\Delta^{(2)}]^{ab}_{rsuv}$. Apart from the graph including the mass insertion, current conservation guarantees again that the result is independent of x_0, y_0.

Once the flavour structure is projected onto the 27-plet according to appendix A, we get our final result,

$$[\mathcal{C}_{27}]^{ab}_{rsuv}(x_0, y_0) \equiv \int_{\mathbf{x}} \int_{\mathbf{y}} \langle \mathcal{J}^a_0(x) [\hat{\mathcal{O}}_w]^+_{rsuv}(0) \mathcal{J}^b_0(y) \rangle \,. \tag{4.11}$$

It is directly obtained from eq. (4.10), by simply making the replacements

$$[\Delta^{(i)}]^{ab}_{rsuv} \to \hat{\Delta}^{ab}_{rsuv} \,, \qquad i = 1, 2 \,, \tag{4.12}$$

where

$$\begin{aligned} \hat{\Delta}^{ab}_{rsuv} \equiv \frac{1}{2}\Big(T^{\{a}_{us}T^{b\}}_{vr} + T^{\{a}_{ur}T^{b\}}_{vs}\Big) + \frac{1}{20}\Big(\delta_{us}\delta_{vr} + \delta_{ur}\delta_{vs}\Big)\,\mathrm{Tr}\; T^a T^b - \\ - \frac{1}{10}\Big(\delta_{us}\{T^a, T^b\}_{vr} + \delta_{vr}\{T^a, T^b\}_{us} + \delta_{ur}\{T^a, T^b\}_{vs} + \delta_{vs}\{T^a, T^b\}_{ur}\Big) \,. \end{aligned} \tag{4.13}$$

As a simple explicit example, we again assume a box with $L_0 = L_1 = L_2 = L_3$, a mass matrix $M = \mathrm{diag}(m, m, m)$, a phase $\theta = 0$, $N_f = 3$, and choose generators such that

$$T^a_{ij} \equiv \delta_{iu}\delta_{js} \,, \qquad T^b_{ij} \equiv \delta_{id}\delta_{ju} \,. \tag{4.14}$$

We also choose the physical indices for $[\hat{\mathcal{O}}_w]^+_{rsuv}$, according to eq. (2.17). Then $\hat{\Delta}^{ab}_{suud} = 2/5$, and the full result is

$$[\mathcal{C}_{27}]^{ab}_{suud}(x_0, y_0) = -\frac{F_\pi^4}{4L_0^2}\frac{2}{5}\bigg\{1 + \frac{1}{F_\pi^2 L_0^2}\bigg[4\beta_1 + u\,\Sigma_{\theta=0}\Big(\frac{u}{2}\Big)\Big[h_1\Big(\frac{x_0}{L_0}\Big) + h_1\Big(\frac{y_0}{L_0}\Big)\Big]\bigg]\bigg\} \,, \tag{4.15}$$

where the notation is as in eq. (4.4).

In figure 4 we show the predictions of eq. (4.10) for this index choice, normalised to the tree-level value, for two asymmetric volumes. Since there is only one 27-plet operator in eq. (2.7), a measurement of g_{27} can then be obtained through the matching of the chiral three-point function with the corresponding lattice QCD measurement:

$$\frac{5}{3} g_{27} [\mathcal{C}_{27}]^{ab}_{suud}(x_0, y_0) = h_w^+ \int_{\mathbf{x}} \int_{\mathbf{y}} \langle J^a_0(x) [\hat{\mathcal{O}}_w]^+_{suud}(0) J^b_0(y) \rangle \,. \tag{4.16}$$

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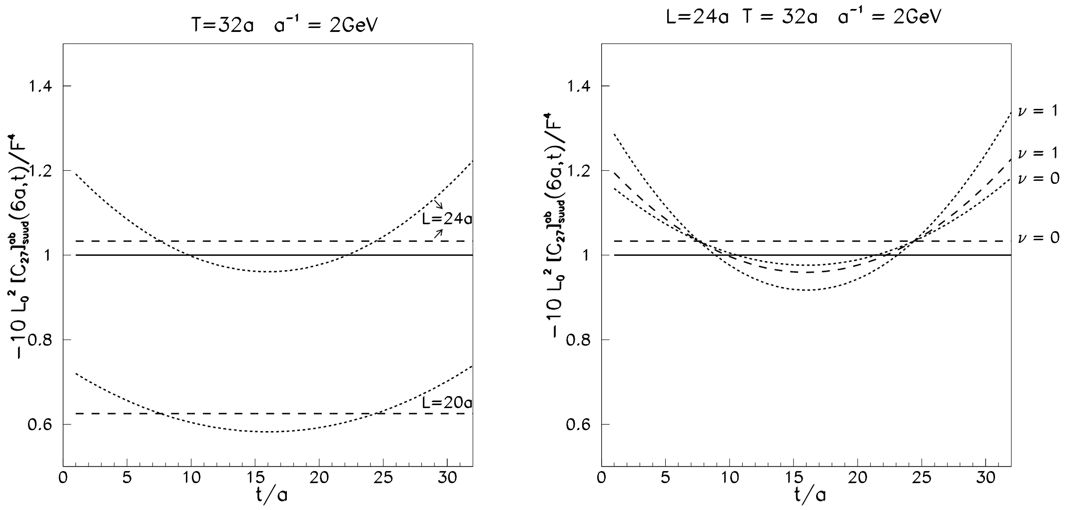

Figure 4: Left: The result of eq. (4.10) for the index choice in eqs. (4.14), (4.15), normalised to the tree-level value (solid line), for the same parameters as in figure 2. The upper set is for $L = 24a$, the lower for $L = 20a$. The dashed and dotted lines correspond to $m = 0\,\mathrm{MeV}$ and $m = 5\,\mathrm{MeV}$, respectively. Right: The same observable, in an ensemble with a fixed topological charge $\nu = 0, 1$ (cf. section 4.3), for $L = 24a$.

4.3 Fixed topology

The results above were obtained in a fixed θ-vacuum. We can also perform a Fourier transform in θ to obtain averages in sectors of "fixed topology" ν [41]. This is interesting because in the quenched theory we expect to find poles in m in fermion propagators, which become dominant in the ϵ-regime when $m\Sigma V \ll 1$. In the θ-vacuum there are no such poles in the full theory, because topological configurations are strongly suppressed by the fermion determinant. However, when considering averages in sectors of non-zero fixed topology, the same poles are expected to appear in the quenched and the full theories. It is quite remarkable that these poles appear also in the corresponding effective chiral theories! Even though their presence does not affect the counting rules of the ϵ-expansion, because $m\Sigma V$ is formally counted as a quantity of $\mathcal{O}(1)$, they obviously modify the chiral limit. The question we want to address here is whether there are such poles in the observables of eqs. (3.2), (3.3).

An observable in the sector of topological charge ν, f_ν, can be obtained from the observable in a θ-vacuum, f_θ, by

$$f_\nu = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta e^{-i\nu\theta} f_\theta\,. \tag{4.17}$$

In particular, assuming again $M = \mathrm{diag}(m, m, m)$ and defining

$$Z_\theta\left(\frac{u}{2}\right) = \int_{U_0} e^{(u/2)\,\mathrm{Re\,Tr}\,[U_0 \exp(i\theta/N_f)]}\,, \qquad Z_\nu\left(\frac{u}{2}\right) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta e^{-i\nu\theta} Z_\theta\left(\frac{u}{2}\right), \tag{4.18}$$

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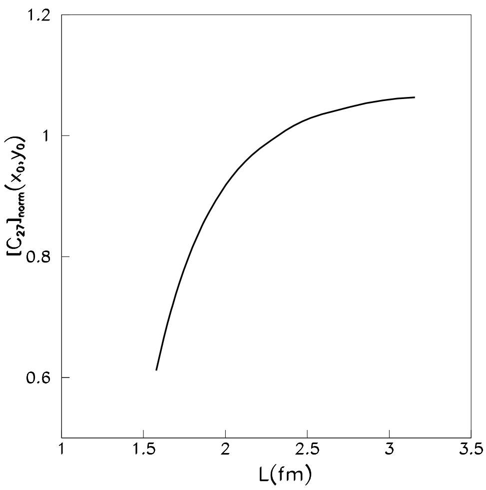

Figure 5: The result of eq. (4.22), for the same parameters as in figure 2. At this order, the outcome is independent of x_0, y_0.

the combination appearing in eqs. (4.1), (4.10) gets replaced as

$$\Sigma_\theta\left(\frac{u}{2}\right) = \frac{2}{N_f}\frac{\partial}{\partial u}\ln Z_\theta\left(\frac{u}{2}\right) \longrightarrow \frac{2}{N_f}\frac{\partial}{\partial u}\ln Z_\nu\left(\frac{u}{2}\right) \equiv \Sigma_\nu\left(\frac{u}{2}\right). \tag{4.19}$$

The function Z_ν is known [42, 41] to be

$$Z_\nu\left(\frac{u}{2}\right) = \det\left[I_{\nu+j-i}\left(\frac{u}{2}\right)\right], \tag{4.20}$$

where the determinant is taken over an $N_f \times N_f$ matrix, whose matrix element (i, j) is the modified Bessel function $I_{\nu+j-i}$.

Thus, at fixed topology the results corresponding to eqs. (4.4), (4.15) are obtained by the substitution $\Sigma_\theta(u/2) \to \Sigma_\nu(u/2)$. For small and large u we have (independent of N_f),

$$\Sigma_\nu\left(\frac{u}{2}\right) \approx \begin{cases} 2|\nu|/u\,, & u \ll 1 \\ 1, & u \gg 1 \end{cases}. \tag{4.21}$$

As expected the low mass behaviour ($u \ll 1$) is drastically modified with respect to that in eq. (4.7). This implies that even though the correlators remain finite for $m \to 0$ (i.e., there are no poles, because $\Sigma_\nu(u/2)$ is multiplied by u), their time dependence does not vanish. This is illustrated for $\mathcal{C}^{ab}(x_0)$ in figure 2 and for $[\mathcal{C}_{27}]^{ab}_{suud}(x_0, y_0)$ in figure 4.

4.4 Normalised correlators

The predictions of the previous sections depend at leading order on the chiral theory parameter F, show rather bad convergence at small volumes, $L \lesssim 2$ fm, and have (for a non-vanishing mass, as well as for a non-zero topological charge) a non-trivial dependence on x_0, y_0. All these dependences are a nuisance for the determination of g_{27}. Fortunately, there seems to be a large cancellation if we normalise the three-point function $[\mathcal{C}_{27}]^{ab}_{suud}$ by two charge – charge correlators, at least at the present order in the ϵ-expansion.

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More precisely, let us again choose the indices in eqs. (4.14), (4.15), and denote by $\mathcal{C}^{aa^\dagger}(x_0)$ a charge — charge correlator obtained by using the generators $T^a, (T^a)^\dagger$ in the currents. Expanding the denominators, we then obtain that to relative order $\mathcal{O}(\epsilon^2)$,

$$[\mathcal{C}_{27}]_{\rm norm}(x_0, y_0) \equiv -\frac{5}{2}\frac{[\mathcal{C}_{27}]^{ab}_{suud}(x_0, y_0)}{\mathcal{C}^{aa^\dagger}(x_0)\mathcal{C}^{bb^\dagger}(y_0)} = 1 + \frac{2}{F^2}\left[\frac{\beta_1}{V^{1/2}} - \frac{L_0^2 k_{00}}{V}\right]. \qquad (4.22)$$

The same result holds at fixed topology, as defined in section 4.3.

Thus, the time, quark mass, and topology dependences cancel completely in the ratio of eq. (4.22)! The next-to-leading order correction is also numerically smaller than the corresponding corrections in the numerator and denominator separately. This result is illustrated in figure 5 as a function of the spatial volume, for a fixed time-like extent $T = 3.2\,\text{fm}$.

To summarise, the optimal method for determining g_{27} would appear to be from the equality

$$g_{27} = \frac{3}{5} h_w^+ \frac{[C_{27}]_{\rm norm}(x_0, y_0)}{[\mathcal{C}_{27}]_{\rm norm}(x_0, y_0)}\,, \qquad (4.23)$$

where $[C_{27}]_{\rm norm}$ is the QCD-correspondent for the expectation value in eq. (4.22). The independence of the outcome on the volume, quark masses, x_0, y_0, and topological charge, serves as a test of whether the regime of applicability of eq. (4.23) has been reached.

5. The quenched case

5.1 The basic setup

Due to the numerical cost of dynamical Ginsparg-Wilson fermions, practical lattice simulations will, for a while still, have to resort to the quenched approximation. It is therefore of interest to study how the results of the previous sections are expected to be affected by quenching. The tool for this is quenched chiral perturbation theory, applied to the ϵ-regime. Previous results in this setup exist for the quark condensate [43], and the scalar and pseudoscalar [38] as well as flavoured vector and axial-vector [44] two-point functions.

There are two approaches to quenched QCD, believed to be equivalent: the so-called supersymmetric (SUSY) formulation [45, 46] and the so-called replica method [37, 38]. In the former, bosonic "ghost" quarks are introduced in order to cancel the effects of the physical quarks; in the latter, the computation is carried out by keeping separately track of the N_v "valence" quarks appearing in the external sources, and the N_f dynamical quarks, and the quenched limit is obtained by taking $N_f \to 0$ for a fixed $N_v \neq 0$.

The two methods are formally equivalent at the quark level, however their low-energy effective theories appear to be quite different. Assuming that the naive chiral symmetries of these models, $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ in the replica case and the graded $\mathrm{U}(N_v|N_v)_L \times \mathrm{U}(N_v|N_v)_R$ in the SUSY formulation, are broken spontaneously by the formation of a quark condensate to the corresponding vector subgroups, the low-energy degrees of freedom are the resulting Goldstone bosons, whose dynamics can be described by chiral lagrangians, at energies below the typical confinement scale. The field variables of the chiral lagrangians are matrices parametrising the Goldstone manifolds.

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There is one important difference with respect to full QCD, though: the field associated with the singlet axial rotation ($\sim \eta'$) does not decouple from the low-energy dynamics. This is true both in the SUSY method [45, 46], as well as in the replica: in order to have a sensible $N_f \to 0$ limit, the Goldstone manifold needs to be enlarged from $\mathrm{SU}(N_f)$ to $\mathrm{U}(N_f)$. It is then easy to see that the decoupling of the singlet field and the limit $N_f \to 0$ do not commute [37]. As a result, the chiral lagrangian may contain all possible interactions involving the singlet, and to lowest order, the replica method has

$$\mathcal{L}_{\chi PT} = \frac{F^2}{4}\,\mathrm{Tr}\left[\partial_\mu U \partial_\mu U^{-1}\right] - \frac{\Sigma}{2}\,\mathrm{Tr}\left[U_\theta U M + M^\dagger U^{-1} U_\theta^{-1}\right] + \frac{m_0^2}{2N_c}\Phi_0^2 + \frac{\alpha}{2N_c}(\partial_\mu \Phi_0)^2\,, \quad (5.1)$$

where $\Phi_0 \equiv (F/2)\,\mathrm{Tr}\,[-i\ln(U)]$ and $U_\theta \equiv \exp(i\theta I_{N_v}/N_v)$. Here, I_{N_v} is the identity matrix in the valence subspace and zero elsewhere. Obviously the couplings F and Σ need not be the same as in full QCD. In addition, new parameters related to axial singlet field, m_0^2, α, have been introduced. In the SUSY formulation the first order chiral lagrangian is the same, with the substitution $\mathrm{Tr} \;\to\; \mathrm{Str}$ and $U \in \mathrm{U}(N_v|N_v)$ (or, more precisely, $U \in \widehat{\mathrm{Gl}}(N_v|N_v)$ [47]).

Since Φ_0 is a singlet, there could in principle also be additional operators constructed with it in eq. (5.1). They have, however, been shown to be suppressed by additional powers of $1/N_c$ [48], so we will neglect them in the following.

Even though the low-energy lagrangians of the replica and SUSY theories have quite different dynamical degrees of freedom, it is believed that in perturbation theory all computations concerning physical observables are equivalent [37]. In the ϵ-regime, however, a non-perturbative definition is needed for the zero-momentum integration. Traditionally this could only be achieved with the SUSY formulation, but there have been recent developments whereby it is argued that replica integrations can also be performed non-perturbatively, and agree with SUSY integrations [49]. Here we carry out the perturbative part of the computation with the replica method, and return to the zero-mode integrations later on.

Provided that only sectors of fixed topology are considered, the rules of the ϵ-expansion are as in eq. (3.6). The massless non-zero mode Goldstone propagator is also needed. In the replica case, it is given by eq. (3.8), with

$$\begin{aligned} E(x) &= \lim_{N_f\to 0}\int_{p'} \frac{e^{ip\cdot x}}{p^4}\frac{(\alpha p^2 + m_0^2)/(2N_c)}{1 + (N_f/p^2)(\alpha p^2 + m_0^2)/(2N_c)} \\ &\equiv \frac{\alpha}{2N_c}G(x) + \frac{m_0^2}{2N_c}F(x)\,, \qquad F(x) \equiv \int_{p'}\frac{e^{ip\cdot x}}{p^4}\,. \end{aligned} \quad (5.2)$$

However, as mentioned, our previous results were obtained with a completely general $E(x)$, and therefore we know that eqs. (4.1), (4.10) are independent of its form.

5.2 Currents and weak operators

Let us then consider the left-handed currents, in the replica formulation. Since the current follows from the lagrangian, cf. eq. (2.14), and the additional degree of freedom Φ_0 is a

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flavour singlet, nothing changes with respect to the unquenched case at the present order:

$$\mathcal{J}_\mu^{a,\text{quenched}} = -i\frac{F^2}{2}T^a_{rs}\Big(\partial_\mu U U^{-1}\Big)_{sr}\,, \tag{5.3}$$

where the matrices T^a are traceless and take non-zero values only in the valence sector.

For the weak operators which do not directly follow from the lagrangian, we have to be more careful. The general issue is whether there are more operators once the larger symmetry group of the quenched theory is considered. Clearly, for instance, one could attach the singlet field Φ_0 to any operator. As mentioned, we assume such terms to be suppressed by $1/N_c$ and ignore them. Another trivial issue is that trace parts of $(\partial_\mu U U^{-1})_{ru}$ vanish in the full theory but not in the quenched theory. However, there could in principle also be more drastic effects [24].

To be systematic about the operators appearing, let us recall the symmetries that are relevant. The weak operators we consider have indices corresponding to left-handed valence flavours only. Therefore they are singlets under the *full* right-handed symmetry group,[5] while they should have the correct symmetry properties under the left-handed *valence* subgroup. These requirements are sufficient to guarantee that the leading order 27-plet operator in the quenched theory is of the same type as in the unquenched case.

Indeed, to get right-handed singlets under the full symmetry group, we are lead to the building blocks

$$\partial_\mu U U^{-1}, U\partial_\mu U^{-1} \ \sim \mathcal{O}(\epsilon^2)\,; \qquad UM,\ M^\dagger U^{-1} \sim \mathcal{O}(\epsilon^4)\,, \tag{5.4}$$

which transform as fundamental $\otimes$ anti-fundamental under the valence subgroup $\mathrm{SU}(N_v)_L$, or $\mathrm{SU}(N_v|N_v)_L$. We have also indicated the scalings of these operators in the ϵ-regime. The operators can be trivially decomposed into a sum of $\mathbf{3}\otimes\mathbf{3}^*$, $\mathbf{1}\otimes\mathbf{3}^*$, $\mathbf{3}\otimes\mathbf{1}$ and $\mathbf{1}\otimes\mathbf{1}$ irreducible representations of the valence subgroup. To get a Lorentz invariant object with four flavour indices leads, as in the unquenched case, to a unique possibility up to and including $\mathcal{O}(\epsilon^6)$:

$$[\mathcal{O}_w]_{rsuv} = \frac{1}{4}F^4\Big(\partial_\mu U U^{-1}\Big)_{ur}\Big(\partial_\mu U U^{-1}\Big)_{vs}\,. \tag{5.5}$$

The reduction to irreducible representations follows from appendix A. Only the $\mathbf{3}\otimes\mathbf{3}^*$ components of $\partial_\mu U U^{-1}$ contribute to the 27-plet, because the 27-plet cannot appear in the tensor product of less than four fundamentals/anti-fundamentals. In other words, the operator $[\hat{\mathcal{O}}_w]^+_{rsuv}$ is symmetric in $r \leftrightarrow s$ and $u \leftrightarrow v$ and traceless in the $\mathrm{SU}(3)_L$ subgroup, and zero if any of the indices lies outside of this subgroup. The weak hamiltonian reads then

$$\mathcal{H}_w = 2\sqrt{2}G_F V_{ud}V^*_{us}\bigg\{\frac{5}{3}g_{27}^{\text{quenched}}[\hat{\mathcal{O}}_w]^+_{suud} + \cdots\bigg\} + \text{h.c.}\,, \tag{5.6}$$

where we have indicated that the quenched coupling, g_{27}^{quenched}, does not need to be the same as the analogous one in the unquenched theory.

[5]This excludes the problematic operators considered in ref. [24], $\sim [U\,\mathrm{diag}(I_{N_v}, -I_{N_v})\,U^{-1}]$, written here in the SUSY formulation.

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In the case of the octets, on the other hand, the classification according to the valence group leads to ambiguities as discussed in [24].

Given that the currents have the same form as in the full theory, the quenched two-point function $\mathcal{C}^{ab,\text{quenched}}$ can now be obtained by setting $N_f \to 0$ in the fixed-topology version of eq. (4.1), as this result is independent of the part $E(x)$ of the propagator. Alternatively, the result can be read from [44], by adding up the vector and axial-vector correlators and dividing by four:

$$\mathcal{C}^{ab,\text{quenched}}(x_0) = \Big(-\operatorname{Tr} T^a T^b\Big)\frac{F^2}{2L_0} \times \left\{1 + \frac{2\Sigma L_0^2}{F^2}\frac{1}{N_v}\Big\langle \operatorname{Re}\operatorname{Tr}_v\,[MU_0]\Big\rangle_{\nu,U_0} h_1\Big(\frac{x_0}{L_0}\Big)\right\}, \tag{5.7}$$

where Tr_v denotes the trace over the valence subgroup, with $N_v = 3$. Let us stress that the absence of $E(x)$ guarantees that there is no dependence on the singlet couplings m_0^2 and α, which appear in the non-zero mode propagator of eq. (5.2). Another interesting point to note is that the terms proportional to N_f of the unquenched result were largely responsible for the large corrections in realistic volumes, while they are absent now. This seems to imply that the ϵ-expansion converges better in the quenched case.

Similarly, the result for the three-point function can be directly extracted from the fixed-topology version of eq. (4.10):

$$\begin{aligned}[\mathcal{C}_w]^{ab,\text{quenched}}_{rsuv}(x_0,y_0) = -\frac{F^4}{4L_0^2}\Bigg\{&\Delta^{(1)} + \Delta^{(2)}\frac{2}{F^2}\left[\frac{\beta_1}{V^{1/2}} - \frac{L_0^2 k_{00}}{V}\right] + \\ &+\Delta^{(1)}\frac{2\Sigma L_0^2}{F^2}\frac{1}{N_v}\Big\langle \operatorname{Re}\operatorname{Tr}_v\,[MU_0]\Big\rangle_{\nu,U_0}\left[h_1\Big(\frac{x_0}{L_0}\Big)+h_1\Big(\frac{y_0}{L_0}\Big)\right]\Bigg\},\end{aligned} \tag{5.8}$$

where we have omitted the indices from $[\Delta^{(1)}]^{ab}_{rsuv}, [\Delta^{(2)}]^{ab}_{rsuv}$, given in eqs. (4.8), (4.9). Again, the unphysical axial singlet couplings m_0^2, α do not appear at this order.

An important point to stress is that the zero-mode integrals appearing in eqs. (5.7), (5.8) are identical. Therefore, if we make the index choice of eqs. (4.14), (4.15) and consider the ratio of eq. (4.22), the results are the same in the unquenched and quenched cases:

$$\begin{aligned}[\mathcal{C}_{27}]^{\text{quenched}}_{\text{norm}}(x_0,y_0) &\equiv -\frac{5}{2}\frac{[\mathcal{C}_{27}]^{ab,\text{quenched}}_{suud}(x_0,y_0)}{\mathcal{C}^{aa^\dagger,\text{quenched}}(x_0)\mathcal{C}^{bb^\dagger,\text{quenched}}(y_0)} \\ &= 1 + \frac{2}{F^2}\left[\frac{\beta_1}{V^{1/2}} - \frac{L_0^2 k_{00}}{V}\right] + \mathcal{O}(\epsilon^4)\,.\end{aligned} \tag{5.9}$$

5.3 The quenched chiral condensate

We end by discussing the actual value of the zero-mode integral in eqs. (5.7), (5.8) for degenerate masses $M = \operatorname{diag}(m,m,m)$:

$$\Sigma_\nu^{\text{quenched}} \equiv \frac{2}{N_v}\left\langle \operatorname{Tr}_v\,[U_0 + U_0^{-1}]\right\rangle_{\nu,U_0}\,. \tag{5.10}$$

This is just the quark condensate, and the value is well known, for $N_v = 1$ [43]. It is usually assumed that the outcome should not depend on $N_v \geq 1$. What we wish to do here is to

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check this explicitly for $N_v = 2, 3$, by using the recent results from [49]:

$$Z_{\nu,N_v}[J] \equiv \lim_{N_f\to 0} \int_{U_0\,\in\,\mathrm{U}(N_f)} \det{}^\nu U_0 e^{\Sigma V\,\mathrm{Re\,Tr}[M_J U_0]} \sim \frac{\det\left[\mathcal{I}_{\nu,ij}(\mu_i)\right]_{i,j=1,\ldots,2N_v}}{\prod_{j>i=1}^{N_v}(\mu_j^2-\mu_i^2)}\,, \tag{5.11}$$

where the $N_f \times N_f$ matrix M_J is $M_J \equiv \mathrm{diag}(m + J_1, \ldots, m + J_{N_v}, m, \ldots)$, and $\mu_i = (m + J_i)\Sigma V$, $\mu = m\Sigma V$, together with

$$\mathcal{I}_{\nu,ij}(\mu_i) \equiv \begin{cases} \mu_i^{j-1} I_{\nu+j-1}(\mu_i), & i = 1, \ldots, N_v\,, \\ (-1)^{j-i+N_\nu}\mu^{j-1}K_{\nu+j-i+N_\nu}(\mu), & i = N_v+1, \ldots, 2N_v\,. \end{cases} \tag{5.12}$$

The derivative of the logarithm of this partition function with respect to J_1, evaluated at $J_i = 0$, gives the required integral in eq. (5.10) for any N_v. The result indeed agrees for $N_v = 2, 3$ with the SUSY result obtained with U(1|1) [43]:

$$\Sigma_\nu^{\mathrm{quenched}} = \mu\Big[I_\nu(\mu)K_\nu(\mu) + I_{\nu+1}(\mu)K_{\nu-1}(\mu)\Big] + \frac{\nu}{\mu}\,. \tag{5.13}$$

For $\nu \neq 0$, the leading behaviour of this function at small and large mass is the same as in the unquenched case at fixed topology, eq. (4.21):

$$\Sigma_\nu^{\mathrm{quenched}}(\mu) \approx \begin{cases} |\nu|/\mu, \; \mu \ll 1 \\ \;\;1, \quad\; \mu \gg 1 \end{cases}. \tag{5.14}$$

6. Conclusions

We have computed the two-point correlator of left-handed chiral charges, as well as the three-point correlator of two left-handed charges and one strangeness violating weak operator [13], in SU(3) chiral perturbation theory in a finite volume and close to the chiral limit, at next-to-leading order in the ϵ-expansion. The comparison of these observables with lattice data would in principle permit the extraction of the pion decay constant F, as well as the low-energy constant g_{27}, involved in the $\Delta I = 3/2$ kaon decays and in the kaon mixing parameter $\hat{B}_K$, with a minimal contamination from higher order corrections. Whether this will be numerically feasible is still an open question.

We have also performed the same calculations in the quenched theory, using its replica formulation, and shown that these observables are only moderately affected. In particular, the ratio defined in eq. (4.22) is not only independent of the quark masses sufficiently close to the chiral limit, but also unaffected by quenching at this order. Obviously the low-energy constants obtained with the quenched theory nevertheless differ from those in full QCD.

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A. SU(3) classification

For completeness, we reiterate in this appendix some essential aspects of the SU(3) classification of four quark operators. We follow the tensor method discussed, e.g., in [50].

The tensors we need to consider are of the form O_{rsuv}, symmetric under $(r \leftrightarrow s, u \leftrightarrow v)$, and transforming under $\mathbf{3}^* \otimes \mathbf{3}^* \otimes \mathbf{3} \otimes \mathbf{3}$ of SU(3). We then define the projected operators

$$O^{\sigma}_{rsuv} \equiv (P^{\sigma}_1)_{rsuv;\tilde{r}\tilde{s}\tilde{u}\tilde{v}}\; O_{\tilde{r}\tilde{s}\tilde{u}\tilde{v}} \,, \tag{A.1}$$

$$\hat{O}^{\sigma}_{rsuv} \equiv (P^{\sigma}_2)_{rsuv;\tilde{r}\tilde{s}\tilde{u}\tilde{v}}\; O^{\sigma}_{\tilde{r}\tilde{s}\tilde{u}\tilde{v}} \,, \tag{A.2}$$

where $\sigma = \pm 1$. Here, with some redundancy in the symmetries of P^{σ}_1,

$$(P^{\sigma}_1)_{rsuv;\tilde{r}\tilde{s}\tilde{u}\tilde{v}} \equiv \frac{1}{4}(\delta_{r\tilde{r}}\delta_{s\tilde{s}} + \sigma\delta_{r\tilde{s}}\delta_{s\tilde{r}})(\delta_{u\tilde{u}}\delta_{v\tilde{v}} + \sigma\delta_{u\tilde{v}}\delta_{v\tilde{u}}) \,, \tag{A.3}$$

$$\begin{aligned}(P^{\sigma}_2)_{rsuv;\tilde{r}\tilde{s}\tilde{u}\tilde{v}} \equiv{}& \delta_{r\tilde{r}}\delta_{s\tilde{s}}\delta_{u\tilde{u}}\delta_{v\tilde{v}} + \frac{1}{(N_f+2\sigma)(N_f+\sigma)}(\delta_{ru}\delta_{sv} + \sigma\delta_{rv}\delta_{su})\delta_{\tilde{r}\tilde{u}}\delta_{\tilde{s}\tilde{v}} - \frac{1}{N_f+2\sigma} \times \\ &\times (\delta_{ru}\delta_{s\tilde{s}}\delta_{v\tilde{v}}\delta_{\tilde{r}\tilde{u}} + \delta_{sv}\delta_{r\tilde{r}}\delta_{u\tilde{u}}\delta_{\tilde{s}\tilde{v}} + \sigma\delta_{rv}\delta_{s\tilde{s}}\delta_{u\tilde{v}}\delta_{\tilde{r}\tilde{u}} + \sigma\delta_{su}\delta_{r\tilde{r}}\delta_{v\tilde{u}}\delta_{\tilde{s}\tilde{v}}) \,,\end{aligned} \tag{A.4}$$

where $N_f = 3$.

It is easy to see that the antisymmetric tensor $\hat{O}^{-}_{rsuv}$ vanishes identically. The reason is that (as can be understood for instance by contracting with $\epsilon_{krs}\epsilon_{luv}$) it corresponds to a representation with dimension 8 just like R^{-}_{lk} defined in eq. (A.6), but all such representations have already been subtracted by the projection operator in eq. (A.4).

Consequently, the reduction of a general operator O_{rsuv} proceeds as

$$\begin{aligned}O_{rsuv} = \hat{O}^{+}_{rsuv} + \sum_{\sigma=\pm1}\Bigg[&\frac{1}{3(3+\sigma)}(\delta_{ru}\delta_{sv} + \sigma\delta_{rv}\delta_{su})S^{\sigma} + \\ &+ \frac{1}{3+2\sigma}(\delta_{ru}R^{\sigma}_{sv} + \delta_{sv}R^{\sigma}_{ru} + \sigma\delta_{rv}R^{\sigma}_{su} + \sigma\delta_{su}R^{\sigma}_{rv})\Bigg] \,,\end{aligned} \tag{A.5}$$

where $\hat{O}^{+}_{rsuv}$ transforms under the representation with dimension 27, and

$$\begin{aligned}S^{\sigma} &\equiv O^{\sigma}_{klkl} \,, \\ R^{\sigma}_{ru} &\equiv O^{\sigma}_{rkuk} - \frac{1}{3}\delta_{ru}S^{\sigma} \,.\end{aligned} \tag{A.6}$$

Here $R^{\pm}_{ru}$'s have the dimension 8, while S^{σ} are singlets.

Finally, let us note that in the chiral theory, i.e. if we replace $O_{rsuv} \to [\mathcal{O}_w]_{rsuv}$,

$$[\mathcal{O}_w]_{rkuk} = [\mathcal{O}_w]_{kskv} = 0 \,, \tag{A.7}$$

so that

$$[\mathcal{O}_w]^{\sigma}_{rkuk} = \frac{\sigma}{2}[\mathcal{O}_w]_{rkku} \,, \qquad [\mathcal{S}_w]^{\sigma} = \frac{\sigma}{2}[\mathcal{O}_w]_{lkkl} \,, \qquad [\mathcal{R}_w]^{+}_{ru} = -[\mathcal{R}_w]^{-}_{ru} \,. \tag{A.8}$$

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References

[1] N. Cabibbo, G. Martinelli and R. Petronzio, *Weak interactions on the lattice*, *Nucl. Phys.* **B 244** (1984) 381;
R.C. Brower, G. Maturana, M. Belen Gavela and R. Gupta, *Calculation of weak transitions in lattice QCD*, *Phys. Rev. Lett.* **53** (1984) 1318.

[2] C.W. Bernard, T. Draper, A. Soni, H.D. Politzer and M.B. Wise, *Application of chiral perturbation theory to $K \to 2\pi$ decays*, *Phys. Rev.* **D 32** (1985) 2343.

[3] P.H. Ginsparg and K.G. Wilson, *A remnant of chiral symmetry on the lattice*, *Phys. Rev.* **D 25** (1982) 2649.

[4] D.B. Kaplan, *A method for simulating chiral fermions on the lattice*, *Phys. Lett.* **B 288** (1992) 342 [hep-lat/9206013].

[5] Y. Shamir, *Chiral fermions from lattice boundaries*, *Nucl. Phys.* **B 406** (1993) 90 [hep-lat/9303005];
V. Furman and Y. Shamir, *Axial symmetries in lattice QCD with Kaplan fermions*, *Nucl. Phys.* **B 439** (1995) 54 [hep-lat/9405004].

[6] R. Narayanan and H. Neuberger, *Chiral determinant as an overlap of two vacua*, *Nucl. Phys.* **B 412** (1994) 574 [hep-lat/9307006]; *A construction of lattice chiral gauge theories*, *Nucl. Phys.* **B 443** (1995) 305 [hep-th/9411108].

[7] H. Neuberger, *Exactly massless quarks on the lattice*, *Phys. Lett.* **B 417** (1998) 141 [hep-lat/9707022]; *More about exactly massless quarks on the lattice*, *Phys. Lett.* **B 427** (1998) 353 [hep-lat/9801031]; *Vector like gauge theories with almost massless fermions on the lattice*, *Phys. Rev.* **D 57** (1998) 5417 [hep-lat/9710089].

[8] P. Hasenfratz, *Lattice QCD without tuning, mixing and current renormalization*, *Nucl. Phys.* **B 525** (1998) 401 [hep-lat/9802007].

[9] M. Lüscher, *Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation*, *Phys. Lett.* **B 428** (1998) 342 [hep-lat/9802011].

[10] Y. Kikukawa and T. Noguchi, *Low energy effective action of domain-wall fermion and the Ginsparg-Wilson relation*, hep-lat/9902022.

[11] S. Capitani, M. Göckeler, R. Horsley, P.E.L. Rakow and G. Schierholz, *Operator improvement for Ginsparg-Wilson fermions*, *Phys. Lett.* **B 468** (1999) 150 [hep-lat/9908029];
S. Aoki and Y. Kuramashi, *Perturbative renormalization factors of $\Delta S = 1$ four-quark operators for domain-wall QCD*, *Phys. Rev.* **D 63** (2001) 054504 [hep-lat/0007024].

[12] C. Alexandrou, E. Follana, H. Panagopoulos and E. Vicari, *One-loop renormalization of fermionic currents with the overlap-Dirac operator*, *Nucl. Phys.* **B 580** (2000) 394 [hep-lat/0002010];
S. Capitani and L. Giusti, *Perturbative renormalization of weak-hamiltonian four-fermion operators with overlap fermions*, *Phys. Rev.* **D 62** (2000) 114506 [hep-lat/0007011];
S. Capitani and L. Giusti, *Analysis of the $\Delta I = 1/2$ rule and ϵ'/ϵ with overlap fermions*, *Phys. Rev.* **D 64** (2001) 014506 [hep-lat/0011070].

[13] L. Giusti, C. Hoelbling, M. Lüscher and H. Wittig, *Numerical techniques for lattice QCD in the ϵ-regime*, hep-lat/0212012.

JHEP01(2003)063

[14] L. Lellouch and M. Lüscher, *Weak transition matrix elements from finite-volume correlation functions*, *Commun. Math. Phys.* **219** (2001) 31 [hep-lat/0003023].

[15] J. Gasser and H. Leutwyler, *Thermodynamics of chiral symmetry*, *Phys. Lett.* **B 188** (1987) 477;
Spontaneously broken symmetries: effective lagrangians at finite volume, *Nucl. Phys.* **B 307** (1988) 763.

[16] H. Neuberger, *A better way to measure F_π in the linear sigma model*, *Phys. Rev. Lett.* **60** (1988) 889; *Soft pions in large boxes*, *Nucl. Phys.* **B 300** (1988) 180.

[17] F.C. Hansen, *Finite size effects in spontaneously broken* $\mathrm{SU}(N) \times \mathrm{SU}(N)$ *theories*, *Nucl. Phys.* **B 345** (1990) 685.

[18] F.C. Hansen and H. Leutwyler, *Charge correlations and topological susceptibility in QCD*, *Nucl. Phys.* **B 350** (1991) 201.

[19] J.A. Cronin, *Phenomenological model of strong and weak interactions in chiral* $\mathrm{U}(3) \times \mathrm{U}(3)$, *Phys. Rev.* **161** (1967) 1483.

[20] N. Ishizuka, *$K \to \pi\pi$ decay amplitude on the lattice*, hep-lat/0209108.

[21] C. Bernard, S. Hashimoto, D.B. Leinweber, P. Lepage, E. Pallante, S.R. Sharpe and H. Wittig, *Panel discussion on chiral extrapolation of physical observables*, hep-lat/0209086.

[22] J. Kambor, J. Missimer and D. Wyler, *The chiral loop expansion of the nonleptonic weak interactions of mesons*, *Nucl. Phys.* **B 346** (1990) 17;
G. Ecker, J. Kambor and D. Wyler, *Resonances in the weak chiral lagrangian*, *Nucl. Phys.* **B 394** (1993) 101.

[23] SPQ(CD)R collaboration, P. Boucaud et al., *$K \to \pi\pi$ matrix elements beyond the leading-order chiral expansion*, *Nucl. Phys.* **106** *(Proc. Suppl.)* (2002) 329 [hep-lat/0110206];
J. Laiho and A. Soni, *On lattice extraction of $K \to \pi\pi$ amplitudes to $O(p^4)$ in chiral perturbation theory*, *Phys. Rev.* **D 65** (2002) 114020 [hep-ph/0203106];
C.J.D. Lin, G. Martinelli, E. Pallante, C.T. Sachrajda and G. Villadoro, *$K^+ \to \pi^+\pi^0$ decays on finite volumes and at next-to-leading order in the chiral expansion*, hep-lat/0208007;
C.J.D. Lin, G. Martinelli, E. Pallante, C.T. Sachrajda and G. Villadoro, *Finite-volume two-pion amplitudes in the $I = 0$ channel*, *Phys. Lett.* **B 553** (2003) 229 [hep-lat/0211043].

[24] M. Golterman and E. Pallante, *Effects of quenching and partial quenching on penguin matrix elements*, *J. High Energy Phys.* **10** (2001) 037 [hep-lat/0108010];
On the effects of (partial) quenching on penguin contributions to $K \to \pi\pi$, hep-lat/0212008.

[25] H. Georgi, *Weak interactions and modern particle theory*, Benjamin/Cummings, Menlo Park, California 1984;
J.F. Donoghue, E. Golowich and B.R. Holstein, *Dynamics of the standard model*, Cambridge University Press, Cambridge 1992.

[26] A. Pich, *Chiral perturbation theory*, *Rept. Prog. Phys.* **58** (1995) 563 [hep-ph/9502366];
G. Ecker, *Low-energy QCD*, *Prog. Part. Nucl. Phys.* **36** (1996) 71 [hep-ph/9511412].

[27] M.K. Gaillard and B.W. Lee, *$\Delta I = 1/2$ rule for nonleptonic decays in asymptotically free field theories*, *Phys. Rev. Lett.* **33** (1974) 108;
G. Altarelli and L. Maiani, *Octet enhancement of nonleptonic weak interactions in asymptotically free gauge theories*, *Phys. Lett.* **B 52** (1974) 351;

JHEP01(2003)063

G. Altarelli, G. Curci, G. Martinelli and S. Petrarca, *QCD nonleading corrections to weak decays as an application of regularization by dimensional reduction*, *Nucl. Phys.* **B 187** (1981) 461;
A.J. Buras and P.H. Weisz, *QCD nonleading corrections to weak decays in dimensional regularization and 't Hooft-Veltman schemes*, *Nucl. Phys.* **B 333** (1990) 66.

[28] A. Pich, B. Guberina and E. de Rafael, *Problem with the $\Delta I = 1/2$ rule in the standard model*, *Nucl. Phys.* **B 277** (1986) 197;
A. Pich and E. de Rafael, *Four quark operators and nonleptonic weak transitions*, *Nucl. Phys.* **B 358** (1991) 311.

[29] R.J. Crewther, *Chiral reduction of $K \to 2\pi$ amplitudes*, *Nucl. Phys.* **B 264** (1986) 277.

[30] J. Gasser and U.G. Meissner, *On the phase of ϵ'*, *Phys. Lett.* **B 258** (1991) 219.

[31] J. Kambor, J. Missimer and D. Wyler, *$K \to 2\pi$ and $K \to 3\pi$ decays in next-to-leading order chiral perturbation theory*, *Phys. Lett.* **B 261** (1991) 496;
J. Kambor, J.F. Donoghue, B.R. Holstein, J. Missimer and D. Wyler, *Chiral symmetry tests in nonleptonic K decay*, *Phys. Rev. Lett.* **68** (1992) 1818;
J. Bijnens, E. Pallante and J. Prades, *Obtaining $K \to \pi\pi$ from off-shell $K \to \pi$ amplitudes*, *Nucl. Phys.* **B 521** (1998) 305 [hep-ph/9801326];
E. Pallante, *The generating functional for hadronic weak interactions and its quenched approximation*, *J. High Energy Phys.* **01** (1999) 012 [hep-lat/9808018].

[32] E. Pallante and A. Pich, *Strong enhancement of ϵ'/ϵ through final state interactions*, *Phys. Rev. Lett.* **84** (2000) 2568 [hep-ph/9911233]; *Final state interactions in kaon decays*, *Nucl. Phys.* **B 592** (2001) 294 [hep-ph/0007208].

[33] J.F. Donoghue, E. Golowich and B.R. Holstein, *The $\Delta S = 2$ matrix element for $K^0\bar{K}^0$ mixing*, *Phys. Lett.* **B 119** (1982) 412.

[34] S. Herrlich and U. Nierste, *The complete $|\Delta S| = 2$ hamiltonian in the next-to-leading order*, *Nucl. Phys.* **B 476** (1996) 27 [hep-ph/9604330].

[35] J. Bijnens and J. Prades, *Scheme dependence of weak matrix elements in the $1/N_c$ expansion*, *J. High Energy Phys.* **01** (2000) 002 [hep-ph/9911392];
S. Peris and E. de Rafael, *$K^0\bar{K}^0$ mixing in the $1/N_c$ expansion*, *Phys. Lett.* **B 490** (2000) 213 [hep-ph/0006146];
E. de Rafael, *Analytic approaches to kaon physics*, hep-ph/0210317.

[36] L. Lellouch, *Light hadron weak matrix elements*, *Nucl. Phys.* **94** *(Proc. Suppl.)* (2001) 142 [hep-lat/0011088];
L. Giusti, *Exact chiral symmetry on the lattice: QCD applications*, hep-lat/0211009.

[37] P.H. Damgaard and K. Splittorff, *Partially quenched chiral perturbation theory and the replica method*, *Phys. Rev.* **D 62** (2000) 054509 [hep-lat/0003017].

[38] P.H. Damgaard, M.C. Diamantini, P. Hernández and K. Jansen, *Finite-size scaling of meson propagators*, *Nucl. Phys.* **B 629** (2002) 445 [hep-lat/0112016].

[39] P. Hasenfratz and H. Leutwyler, *Goldstone boson related finite size effects in field theory and critical phenomena with $O(N)$ symmetry*, *Nucl. Phys.* **B 343** (1990) 241.

[40] M. Lüscher, *Some analytic results concerning the mass spectrum of Yang-Mills gauge theories on a torus*, *Nucl. Phys.* **B 219** (1983) 233.

JHEP01(2003)063

[41] H. Leutwyler and A. Smilga, *Spectrum of Dirac operator and role of winding number in QCD*, *Phys. Rev.* **D 46** (1992) 5607.

[42] R. Brower, P. Rossi and C.-I. Tan, *The external field problem for QCD*, *Nucl. Phys.* **B 190** (1981) 699.

[43] J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, *From chiral random matrix theory to chiral perturbation theory*, *Nucl. Phys.* **B 540** (1999) 317 [hep-th/9806110];
P.H. Damgaard, J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, *The microscopic spectral density of the QCD Dirac operator*, *Nucl. Phys.* **B 547** (1999) 305 [hep-th/9811212].

[44] P.H. Damgaard, P. Hernández, K. Jansen, M. Laine and L. Lellouch, *Finite-size scaling of vector and axial current correlators*, hep-lat/0211020.

[45] C.W. Bernard and M.F.L. Golterman, *Chiral perturbation theory for the quenched approximation of QCD*, *Phys. Rev.* **D 46** (1992) 853 [hep-lat/9204007];
For an introductory review, see M.F.L. Golterman, *Chiral perturbation theory and the quenched approximation of QCD*, *Acta Phys. Polon.* **B 25** (1994) 1731 [hep-lat/9411005].

[46] S.R. Sharpe, *Quenched chiral logarithms*, *Phys. Rev.* **D 46** (1992) 3146 [hep-lat/9205020].

[47] M.R. Zirnbauer, *Riemannian symmetric superspaces and their origin in random matrix theory*, *J. Math. Phys.* **37** (1996) 4986 [math-ph/9808012].

[48] J. Gasser and H. Leutwyler, *Chiral perturbation theory: expansions in the mass of the strange quark*, *Nucl. Phys.* **B 250** (1985) 465.

[49] E. Kanzieper, *Replica field theories, Painlevé transcendents and exact correlation functions*, *Phys. Rev. Lett.* **89** (2002) 250201 [cond-mat/0207745];
K. Splittorff and J.J. Verbaarschot, *Replica limit of the Toda lattice equation*, cond-mat/0209594;
Y.V. Fyodorov and G. Akemann, *Comment on the article* cond-mat/0209594 *by K. Splittorff and J.J.M. Verbaarschot*, cond-mat/0210647.

[50] H. Georgi, *Lie algebras in particle physics, from isospin to unified theories*, Benjamin/Cummings, Reading, Massachusetts 1982.

JHEP01(2003)063